

HARMONIC CONTACT METRIC STRUCTURES, AND SUBMERSIONS

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In memoriam, Professor James Eells, 1926–2007.

ABSTRACT. We study harmonic almost contact structures in the context of contact metric manifolds, and an analysis is carried out when such a manifold fibres over an almost Hermitian manifold, as exemplified by the Boothby-Wang fibration. Two types of almost contact metric warped products are also studied, relating their harmonicity to that of the almost Hermitian structure on the base or fibre.

1. INTRODUCTION

In recent years the study of *harmonic unit vector fields* on Riemannian manifolds has attracted considerable interest: see [6] for a fairly contemporary survey. Such fields are named for the fact that they are harmonic sections of the unit tangent bundle, and as such may be considered “optimal” when compared with nearby unit vector fields. Geometric interest is added by the fact that their total bending is stationary [11]. Although there is no general existence theory for harmonic vector fields, many examples arise as characteristic, or Reeb, fields on contact or almost contact manifolds. In fact contact metric manifolds with harmonic Reeb field ξ have been dubbed *H-contact* in [9], and characterized as those where ξ is an eigenvector of the Ricci operator. They include *K-contact* manifolds [2]. However, when comparing almost contact metric structures one would also like to take into account the geometry of the *hyperplane bundle* $\mathcal{D} \rightarrow M$ orthogonal to the Reeb field, in particular its induced Hermitian structure.

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For this reason, in [10] we introduced the idea of a *harmonic almost contact metric structure* on an orientable Riemannian manifold (M^{2n+1}, g) , defined as follows. Let $N \rightarrow M$ be the fibre bundle with fibre $SO(2n+1)/U(n)$ associated to the principal bundle of g -orthonormal tangent frames of M viz. the odd-dimensional analogue of the twistor bundle in Hermitian geometry. Then an almost contact metric structure on M is parametrized by a unique section σ of N , and is said to be *harmonic* if σ is a harmonic section. In [10] we analysed the geometry of the homogeneous bundle $N \rightarrow M$ and thereby showed that σ is a harmonic section precisely when the following *two* equations are satisfied:

$$\tau(\xi) + \frac{1}{2}JT(\phi) = 0, \quad (1.1)$$

$$\tau(J) = 0. \quad (1.2)$$

Here, ϕ is the fundamental $(1, 1)$ tensor of the almost contact structure, satisfying:

$$\phi^2 = -I + \eta \otimes \xi,$$

where η is the 1-form dual to ξ , and J is the almost complex structure in $\mathcal{D} = \text{im } \phi$ obtained by restricting ϕ . If $\bar{\nabla}$ is the connection in \mathcal{D} obtained by orthogonally projecting the Levi-Civita connection ∇ of (M, g) , then $\tau(\xi)$ and $\tau(J)$ are defined as follows:

$$\tau(\xi) = \nabla^* \nabla \xi - |\nabla \xi|^2 \xi, \quad \tau(J) = [\bar{\nabla}^* \bar{\nabla} J, J],$$

where $\nabla^* \nabla$ and $\bar{\nabla}^* \bar{\nabla}$ are the *rough Laplacians* of (TM, ∇) and $(\mathcal{D}, \bar{\nabla})$, respectively:

$$\nabla^* \nabla \xi = -\text{tr } \nabla^2 \xi = -\nabla_{E_i, E_i}^2 \xi, \quad \bar{\nabla}^* \bar{\nabla} J = -\text{tr } \bar{\nabla}^2 J = -\bar{\nabla}_{E_i, E_i}^2 J,$$

for any local orthonormal frame (E_i) of TM . As will be the case throughout the paper, the summation signs in these formulæ have been omitted. Notice that $\tau(\xi)$ is the \mathcal{D} -component of $\nabla^* \nabla \xi$. Finally, $T(\phi)$ is the following section of \mathcal{D} :

$$T(\phi) = \text{tr}(\bar{\nabla} J \otimes \nabla \xi) = \bar{\nabla}_{E_i} J(\nabla_{E_i} \xi).$$

Equation (1.2) is simply the condition for J to be a harmonic almost complex structure in \mathcal{D} , in the sense of [13]. On the other hand, $\tau(\xi) = 0$ is the equation for ξ to be a

harmonic vector field [15, 16], so (1.1) is perhaps more subtle than one might expect. In particular, an almost contact metric structure with harmonic Reeb field is harmonic only if the additional condition $T(\phi) = 0$ holds. In this paper we refer to equations (1.1) and (1.2) as the *first* and *second harmonic section equations*, respectively.

A general analytic existence theory for equations (1.1) and (1.2) is currently far from being within reach; the best available existence results for harmonic sections only apply to bundles with compact negatively curved fibres [12, 14]. We therefore seek geometric techniques for constructing solutions. In [10] we studied the case where M^{2n+1} is an isometrically immersed hypersurface of an almost Hermitian manifold \tilde{M}^{2n+2} , and the almost contact structure is induced by the ambient almost complex structure. The relationship between equations (1.1) and (1.2) and the almost Hermitian geometry of \tilde{M} is in general rather complicated, but becomes more tractable when \tilde{M} is a Kähler or nearly Kähler manifold, or $\tilde{M} = M \times \mathbb{R}$. In this paper we investigate the ‘dual’ problem, when M^{2n+1} fibres over a $2n$ -dimensional almost Hermitian manifold \hat{M}^{2n} . In the first part of the paper we confine attention to the case when M is a contact metric manifold. We begin §2 with a characterization (Proposition 2.1) of the first harmonic section equation in terms of:

$$h = \frac{1}{2}L_\xi\phi, \quad (1.3)$$

which is an important structural tensor in contact metric geometry [3]. We then note an analogy between contact metric manifolds and $(1, 2)$ -symplectic manifolds in almost Hermitian geometry (Lemma 2.1), which guides us to a characterization of the second harmonic section equation in terms of the **Ricci curvature* $\bar{\rho}^*$ of \mathcal{D} , which is defined:

$$\bar{\rho}^*(X, Y) = g(\bar{R}(X, F_i)JF_i, JY), \quad \text{for all } X, Y \in \mathcal{D}, \quad (1.4)$$

where \bar{R} denotes the curvature of the vector bundle $(\mathcal{D}, \bar{\nabla})$:

$$\bar{R}(X, Y) = \bar{\nabla}_{X,Y}^2 - \bar{\nabla}_{Y,X}^2,$$

and (F_i) is a local orthonormal frame of \mathcal{D} . We show (Proposition 2.2) that the second harmonic section equation is equivalent to J -invariance (or equivalently, symmetry) of $\bar{\rho}^*$. The analogous result in almost Hermitian geometry is that a $(1, 2)$ -symplectic structure is harmonic if and only if its **Ricci curvature* is symmetric [13]. Combining

Propositions 2.1 and 2.2 yields our main result, Theorem 2.1, which is a characterization of harmonic contact metric structures. In particular, an H -contact structure is harmonic if and only if h is co-closed (when viewed as a TM -valued 1-form) and $\bar{\rho}^*$ is symmetric, and a K -contact structure is harmonic precisely when $\bar{\rho}^*$ is symmetric. For practical purposes it is helpful to recast this in terms of the **Ricci curvature of M* , defined:

$$\rho^*(X, Y) = g(R(X, E_i)\phi E_i, \phi Y), \quad \text{for all } X, Y \in TM, \quad (1.5)$$

where R is the Riemann curvature tensor of (M, g) . We then show (Theorem 2.2) that an H -contact structure is harmonic precisely when ρ^* is symmetric, or equivalently ϕ -invariant. As an example, we show (Theorem 2.3) that all contact metric structures satisfying the (κ, μ) -nullity condition of [4] are harmonic. Such structures include the unit tangent bundles of spaces of constant curvature.

In §3 we take the analogy of §2 one step further, to the situation where there exists a Riemannian submersion $\pi: M \rightarrow \hat{M}$ onto an almost Hermitian manifold \hat{M}^{2n} , which intertwines the contact metric structure of M with the almost complex structure of \hat{M} . Then M is necessarily a K -contact manifold and \hat{M} is almost Kähler, and π exists under certain topological conditions, with the classical construction of the Boothby-Wang fibration [5]. We use Theorem 2.1 to prove that the contact metric structure on M is harmonic precisely when the almost Hermitian structure on \hat{M} is harmonic (Theorem 3.1). We then consider the same setup when M is merely an almost contact manifold, in the special case where $M = \hat{M} \times_f \mathbb{R}$, the warped product equipped with the induced almost contact structure. The result here (Theorem 3.2) is that the almost contact structure is harmonic if and only if the almost Hermitian structure is harmonic, provided that the gradient vector ∇f is Kähler null. Finally we consider the reversed warped product $M = \mathbb{R} \times_f \check{M}$, with the almost contact structure induced by an almost Hermitian structure on \check{M}^{2n} , or more generally where M is locally of this form; for example if M is a Kenmotsu manifold, which was studied in [10]. We show (Theorem 3.3) that the almost contact structure is harmonic if and only if the almost Hermitian structure is harmonic, provided the latter is cosymplectic, or the warping function is constant. It is interesting (Propositions 3.1 and 3.3) that in both warped products the Reeb field is harmonic, for any warping function f , which we believe constitutes a new family of examples of harmonic unit vector fields.

2. HARMONIC CONTACT METRIC STRUCTURES

There are a number of useful relations in the tensor algebra of a contact metric manifold, documented in [3]. Firstly:

$$\nabla_\xi \phi = 0, \quad (2.1)$$

from which it follows that ξ is geodesic:

$$\nabla_\xi \xi = 0. \quad (2.2)$$

Also, the tensor h defined in (1.3) is symmetric, trace-free, anti-commutes with ϕ , and verifies:

$$\nabla_X \xi = -\phi X - \phi hX. \quad (2.3)$$

We may regard h as a TM -valued 1-form on M , and form its co-derivative:

$$\delta h = -\nabla_{E_i} h(E_i).$$

Since $h\xi = 0$, and h is symmetric, h is \mathcal{D} -valued. Our first result shows that δh is in fact a section of \mathcal{D} , and appears naturally in the first harmonic section equation. During the proof, and hereafter, we will often denote the Riemannian metric by $\langle \cdot, \cdot \rangle$ instead of g .

Proposition 2.1. *On a contact metric manifold, there is the following identity:*

$$\delta h = \phi \nabla^* \nabla \xi - T(\phi). \quad (2.4)$$

The first harmonic section equation is equivalent to:

$$\tau(\xi) = \phi \delta h. \quad (2.5)$$

The first harmonic section equation is verified on an H -contact manifold if and only if h is co-closed; in particular, it is always verified for a K -contact structure.

Proof. First we recall from [10] the relation:

$$\bar{\nabla} J = \nabla \phi - \langle \nabla \phi, \xi \rangle \xi. \quad (2.6)$$

We also note from (2.2) that:

$$hX = \phi \nabla_X \xi - X, \quad \text{for all } X \in \mathcal{D}. \quad (2.7)$$

Now suppose for convenience that a local orthonormal frame (F_i) of \mathcal{D} has been constructed by $\bar{\nabla}$ -parallel translation of an orthonormal basis of the fibre \mathcal{D}_x of \mathcal{D} over $x \in M$, along radial geodesics; thus $\bar{\nabla} F_i(x) = 0$. Then at x we also have:

$$\begin{aligned} \nabla_{F_i} F_i &= \langle \nabla_{F_i} F_i, \xi \rangle \xi = -\langle F_i, \nabla_{F_i} \xi \rangle \xi = \langle F_i, \phi F_i + \phi h F_i \rangle \xi, \quad \text{by (2.2)} \\ &= -\langle \phi, h \rangle \xi = 0, \end{aligned}$$

since h (resp. ϕ) is symmetric (resp. skew-symmetric). Therefore:

$$\begin{aligned} -\delta h(x) &= \nabla_{F_i} h(F_i) + \nabla_\xi h(\xi) = \nabla_{F_i} (h F_i), \quad \text{since also } h\xi = 0 \text{ and } \xi \text{ is geodesic,} \\ &= \nabla_{F_i} (\phi \nabla_{F_i} \xi), \quad \text{by (2.7)} \\ &= \nabla_{F_i} \phi(\nabla_{F_i} \xi) + \phi \nabla_{F_i} \nabla_{F_i} \xi \\ &= \bar{\nabla}_{F_i} J(\nabla_{F_i} \xi) + \langle \nabla_{F_i} \phi(\nabla_{F_i} \xi), \xi \rangle \xi - \phi \nabla^* \nabla \xi, \quad \text{by (2.6)} \\ &= T(\phi) - \phi \nabla^* \nabla \xi + \langle \nabla_{F_i} \xi, \phi \nabla_{F_i} \xi \rangle \xi, \quad \text{by (2.2)} \\ &= T(\phi) - \phi \nabla^* \nabla \xi, \quad \text{since } \phi \text{ is skew-symmetric.} \end{aligned}$$

This establishes (2.4). Applying ϕ to both sides yields:

$$\phi \delta h = -\tau(\xi) - JT(\phi),$$

which may be used with (1.1) to obtain (2.5). An H -contact structure verifies $\tau(\xi) = 0$, in which case (2.5) is equivalent to $\delta h = 0$. Finally, a K -contact structure is H -contact [9], and is characterized by $h = 0$; hence (2.5) is verified. \square

We now turn to the second harmonic section equation. Our approach is guided by the following key observation, that $\bar{\nabla} J$ is J -anti-invariant.

Lemma 2.1. *In a contact metric manifold, the Hermitian vector bundle $(\mathcal{D}, \bar{\nabla}, J)$ satisfies:*

$$\bar{\nabla}_{JX} J(JY) = -\bar{\nabla}_X J(Y), \quad \text{for all } X, Y \in \mathcal{D}.$$

Proof. The Kähler form for J is the restriction to \mathcal{D} of the 2-form $\Phi(X, Y) = g(X, \phi Y)$. Suppose $X, Y, Z \in \mathcal{D}$. Then:

$$\begin{aligned} d\Phi(X, Y, Z) &= \nabla_X \Phi(Y, Z) + \nabla_Y \Phi(Z, X) + \nabla_Z \Phi(X, Y) \\ &= \langle Y, \nabla_X \phi(Z) \rangle + \langle Z, \nabla_Y \phi(X) \rangle + \langle X, \nabla_Z \phi(Y) \rangle \\ &= \langle Y, \bar{\nabla}_X J(Z) \rangle + \langle Z, \bar{\nabla}_Y J(X) \rangle + \langle X, \bar{\nabla}_Z J(Y) \rangle, \quad \text{by (2.6).} \end{aligned}$$

One can then establish the following (remarkable) identity, which we leave as an exercise for the reader:

$$\begin{aligned} d\Phi(X, Y, Z) - d\Phi(X, JY, JZ) + d\Phi(JX, JY, Z) + d\Phi(JX, Y, JZ) \\ = -2\langle \bar{\nabla}_X J(Y) + \bar{\nabla}_{JX} J(JY), Z \rangle. \end{aligned}$$

The result follows since $\Phi = d\eta$ for a contact metric structure; thus $d\Phi = 0$. \square

Lemma 2.1 may also be derived from Lemma 7.3 of [3]; however, Lemma 2.1 is *not* equivalent to the ϕ -anti-invariance of $\nabla\phi$.

It follows from (2.1) and (2.6) that in addition to the relation of Lemma 2.1 we have:

$$\bar{\nabla}_\xi J = 0. \quad (2.9)$$

Contact metric manifolds may therefore be regarded as odd-dimensional analogues of $(1, 2)$ -symplectic manifolds. It should be noted however that this analogy is not a characterization. For example, the Kenmotsu almost contact metric structure on the warped product $M = \mathbb{R} \times_f \check{M}^{2n}$, where $f(t) = e^t$ and \check{M} is a Kähler manifold, satisfies $\bar{\nabla}J = 0$, but is not a contact structure since $\mathcal{D} = T\check{M}$ which is clearly integrable. We will see other examples of this in §3.

In order to derive a characterization of the harmonicity of the almost complex structure J analogous to that of [13] for $(1, 2)$ -symplectic structures, we now prove a technical lemma, which introduces the curvature tensor.

Lemma 2.2. *Let F be an element of \mathcal{D} , the contact subbundle of a contact metric manifold. Then:*

$$[\bar{\nabla}_{F,F}^2 J - 2\bar{R}(F, JF) + \bar{\nabla}_{JF,JF}^2 J, J] = 4\bar{\nabla}_{\bar{\nabla}J(F,F)} J.$$

Proof. Suppose that $F, X \in \mathcal{D}_x$, with extensions to local sections of \mathcal{D} such that $\bar{\nabla}X(x) = 0 = \bar{\nabla}F(x)$. Note first that in this case, since the Levi-Civita connection is torsion free, we get using (2.3):

$$\begin{aligned}
[F, JF] &= \nabla_F(JF) - \nabla_{JF}F \\
&= \bar{\nabla}_F(JF) + \langle \nabla_F(JF), \xi \rangle \xi - \langle \nabla_{JF}F, \xi \rangle \xi \\
&= \bar{\nabla}_F J(F) - \langle JF, \nabla_F \xi \rangle \xi + \langle F, \nabla_{JF} \xi \rangle \xi \\
&= \bar{\nabla}_F J(F) + \langle JF, JF + JhF \rangle \xi - \langle F, J^2 F + JhJF \rangle \xi \\
&= \bar{\nabla}J(F, F) + 2|F|^2 \xi, \quad \text{since } h \text{ anticommutes with } \phi.
\end{aligned} \tag{2.10}$$

Also, by Lemma 2.1:

$$\bar{\nabla}_{JF}JF = \bar{\nabla}_{JF}J(F) = \bar{\nabla}_F J(JF) = -J \bar{\nabla}J(F, F). \tag{2.11}$$

We then note that:

$$\begin{aligned}
\bar{\nabla}_{F,F}^2 J &= \bar{\nabla}_F \bar{\nabla}_F J - \bar{\nabla}_{\nabla_F F} J = \bar{\nabla}_F \bar{\nabla}_F J - \langle \nabla_F F, \xi \rangle \bar{\nabla}_\xi J \\
&= \bar{\nabla}_F \bar{\nabla}_F J, \quad \text{by (2.9),}
\end{aligned} \tag{2.12}$$

and similarly, using (2.11) and Lemma 2.1:

$$\bar{\nabla}_{JF,JF}^2 J = \bar{\nabla}_{JF} \bar{\nabla}_{JF} J + J \bar{\nabla}_{\bar{\nabla}J(F,F)} J. \tag{2.13}$$

We now begin the main calculation. Using the Leibnitz rule and (2.12), we have:

$$\bar{\nabla}_{F,F}^2 J(JX) = \bar{\nabla}_F \bar{\nabla}_F J(JX) = \bar{\nabla}_F (\bar{\nabla}_F J(JX)) - (\bar{\nabla}J)_{F,F}^2(X),$$

and the first term on the right hand side may be expanded using the curvature tensor as follows:

$$\begin{aligned}
\bar{\nabla}_F (\bar{\nabla}_F J(JX)) &= \bar{\nabla}_F (\bar{\nabla}_{JF} J(X)), \quad \text{by Lemma 2.1} \\
&= \bar{\nabla}_F \bar{\nabla}_{JF} (JX) - \bar{\nabla}_F (J \bar{\nabla}_{JF} X) \\
&= \bar{\nabla}_F \bar{\nabla}_{JF} (JX) - J \bar{\nabla}_F \bar{\nabla}_{JF} X \\
&= \bar{\nabla}_{JF} \bar{\nabla}_F (JX) + \bar{\nabla}_{[F,JF]} (JX) + \bar{R}(F, JF)(JX) \\
&\quad - J \bar{\nabla}_{JF} \bar{\nabla}_F X - J \bar{R}(F, JF)X \\
&= \bar{\nabla}_{JF} \bar{\nabla}_F (JX) - J \bar{\nabla}_{JF} \bar{\nabla}_F X + \bar{\nabla}_{\bar{\nabla}J(F,F)} J(X) + [\bar{R}(F, JF), J]X,
\end{aligned}$$

using (2.10) and (2.9). The calculation continues:

$$\begin{aligned}
& \bar{\nabla}_F(\bar{\nabla}_F J(JX)) - \bar{\nabla}_{\bar{\nabla}J(F,F)}J(X) - [\bar{R}(F, JF), J]X \\
&= \bar{\nabla}_{JF}(\bar{\nabla}_F J(X)) + \bar{\nabla}_{JF}(J \bar{\nabla}_F X) - J \bar{\nabla}_{JF} \bar{\nabla}_F X \\
&= -\bar{\nabla}_{JF}(\bar{\nabla}_{JF} J(JX)), \quad \text{by Lemma 2.1 again,} \\
&= -\bar{\nabla}_{JF} \bar{\nabla}_{JF} J(JX) - \bar{\nabla}_{JF} J(\bar{\nabla}_{JF}(JX)) \\
&= -\bar{\nabla}_{JF, JF}^2 J(JX) + \bar{\nabla}_{\bar{\nabla}J(F,F)}J(X) - (\bar{\nabla}J)_{JF, JF}^2(X), \quad \text{by (2.13).}
\end{aligned}$$

It follows from Lemma 2.1 once again that:

$$(\bar{\nabla}J)_{JF, JF}^2(X) = \bar{\nabla}_F J \circ J \circ \bar{\nabla}_F J(JX) = (\bar{\nabla}J)_{F, F}^2(X).$$

Therefore:

$$\begin{aligned}
& \bar{\nabla}_{F, F}^2 J(JX) + \bar{\nabla}_{JF, JF}^2 J(JX) \\
&= [\bar{R}(F, JF), J]X + 2 \bar{\nabla}_{\bar{\nabla}J(F,F)}J(X) - 2 (\bar{\nabla}J)_{F, F}^2(X). \quad (2.14)
\end{aligned}$$

Since $(\bar{\nabla}J)^2$ commutes with J , whereas $[\bar{R}(F, JF), J]$ anticommutes with J , replacing X by JX in (2.14) and applying J to the resultant equation yields:

$$\begin{aligned}
& J \bar{\nabla}_{F, F}^2 J(X) + J \bar{\nabla}_{JF, JF}^2 J(X) \\
&= -[\bar{R}(F, JF), J]X - 2 \bar{\nabla}_{\bar{\nabla}J(F,F)}J(X) - 2 (\bar{\nabla}J)_{F, F}^2(X). \quad (2.15)
\end{aligned}$$

Subtraction of (2.15) from (2.14) yields the result. \square

For any local orthonormal frame (F_i) of the hyperplane bundle \mathcal{D} , we define analogously to [13]:

$$\bar{\delta}J = -\text{tr}_{\mathcal{D}} \bar{\nabla}J = -\bar{\nabla}_{F_i} J(F_i).$$

Then it is an immediate consequence of Lemma 2.1 that $\bar{\delta}J = 0$ for a contact metric structure. Furthermore, on any almost contact metric manifold, the curvature tensor \bar{R} of \mathcal{D} is related to the \mathcal{D} -component $R_{\mathcal{D}}$ of the curvature tensor R of (M, g) by the following identity [10]:

$$\bar{R}(X, Y)Z = R_{\mathcal{D}}(X, Y)Z + r(\nabla_X \xi, \nabla_Y \xi)Z, \quad (2.16)$$

for all $X, Y, Z \in \mathcal{D}$, where r is the following curvature-type tensor:

$$r(u, v)w = \langle v, w \rangle u - \langle u, w \rangle v. \quad (2.17)$$

Proposition 2.2. *A contact metric structure verifies the second harmonic section equation if and only if the \ast Ricci curvature of \mathcal{D} is symmetric (or, equivalently, J -invariant).*

Proof. We first note the identity:

$$[\bar{\nabla}^* \bar{\nabla} J, J] = -[\bar{R}(F_i, JF_i), J], \quad (2.18)$$

which follows from Lemma 2.2, using $\bar{\delta}J = 0$ and $\bar{\nabla}_{\xi, \xi}^2 J = 0$, where the latter is a consequence of (2.9) and (2.2). Now suppose $Z, W \in \mathcal{D}$. Then, using Bianchi's first identity, on summation:

$$R(F_i, JF_i)Z = -2R(Z, F_i)JF_i. \quad (2.19)$$

Therefore by (2.16) and (2.3), and making extensive use of the symmetry of h and anticommutativity of h and J :

$$\begin{aligned} \langle \bar{R}(F_i, JF_i)Z, JW \rangle &= -2\langle R(Z, F_i)JF_i, JW \rangle - \langle r(JF_i + JhF_i, F_i + hF_i)Z, JW \rangle \\ &= -2\langle R(Z, F_i)JF_i, JW \rangle - 2\langle Z, W \rangle - 2\langle hZ, hW \rangle \\ &= -2\langle \bar{R}(Z, F_i)JF_i, JW \rangle + 2\langle r(Z + hZ, F_i + hF_i)F_i, W \rangle \\ &\quad - 2\langle Z, W \rangle - 2\langle hZ, hW \rangle \\ &= -2\bar{\rho}^*(Z, W) + 4(n-1)\langle Z + hZ, W \rangle - 4\langle hZ, hW \rangle, \end{aligned}$$

where for the final equation we have also used the facts that h is trace-free and $h\xi = 0$.

Therefore, since h is symmetric:

$$\langle [\bar{R}(F_i, JF_i), J](JZ), JW \rangle = 2\bar{\rho}^*(Z, W) - 2\bar{\rho}^*(W, Z),$$

and the result follows from (2.18) and (1.2). \square

It is interesting to note that the characterization of harmonic $(1, 2)$ -symplectic structures in [13] was obtained in a completely different way to Proposition 2.2, using a technique of Lichnerowicz. Combining Propositions 2.1 and 2.1 yields:

Theorem 2.1.

(1) *A contact metric structure is harmonic if and only if $\tau(\xi) = \phi\delta h$ and the \ast Ricci curvature of \mathcal{D} is symmetric.*

- (2) An H -contact structure is harmonic if and only if h is co-closed and the $*$ Ricci curvature of \mathcal{D} is symmetric.
- (3) A K -contact structure is harmonic if and only if the $*$ Ricci curvature of \mathcal{D} is symmetric.

From the proof of Proposition 2.2 it is possible to extract the equation:

$$\begin{aligned}\bar{\rho}^*(Z, W) &= \rho^*(Z, W) + (2n - 1)\langle Z, W \rangle \\ &\quad + 2(n - 1)\langle hZ, W \rangle - 2\langle hZ, hW \rangle,\end{aligned}\tag{2.20}$$

where the $*$ Ricci curvature ρ^* of M is defined in (1.5). Therefore the symmetry of $\bar{\rho}^*$ is equivalent to the symmetry of $\rho^*(\mathcal{D}, \mathcal{D})$. However, whereas $\rho^*(X, \xi) = 0$ for all $X \in TM$, we have:

Lemma 2.3. *On a contact metric manifold:*

$$\rho^*(\xi, Z) = -\langle \delta h, JZ \rangle, \quad \text{for all } Z \in \mathcal{D}.$$

Proof. We first recall the following curvature identity for contact metric manifolds [3, Lemma 7.4]:

$$\langle R(\xi, X)Y, Z \rangle = -\nabla_X \Phi(Y, Z) - \langle X, \nabla_Y(\phi h)Z \rangle + \langle X, \nabla_Z(\phi h)Y \rangle,$$

from which it follows that for all $Z \in \mathcal{D}$ we have:

$$\rho^*(\xi, Z) = -\nabla_{F_i} \Phi(JF_i, JZ) - \langle F_i, \nabla_{JF_i}(\phi h)JZ \rangle + \langle F_i, \nabla_{JZ}(\phi h)JF_i \rangle.$$

Calculating each term on the right hand side in turn:

$$\begin{aligned}\nabla_{F_i} \Phi(JF_i, JZ) &= \langle JF_i, \nabla_{F_i} \phi(JZ) \rangle = -\langle \bar{\nabla}_{F_i} J(JF_i), JZ \rangle, \quad \text{by (2.6)} \\ &= -\langle \bar{\delta} J, Z \rangle = 0, \quad \text{by Lemma 2.1.}\end{aligned}$$

Similarly, using the anticommutativity of h and J :

$$\begin{aligned}\langle F_i, \nabla_{JF_i}(\phi h)JZ \rangle &= \langle F_i, \nabla_{JF_i} \phi(hJZ) + \phi \nabla_{JF_i} h(JZ) \rangle \\ &= -\langle \bar{\nabla}_{F_i} J(F_i), hZ \rangle - \langle \nabla_{JF_i} h(JF_i), JZ \rangle, \quad \text{by (2.6) and Lemma 2.1,} \\ &= \langle \bar{\delta} J, hZ \rangle + \langle \delta h, JZ \rangle = \langle \delta h, JZ \rangle, \quad \text{by Lemma 2.1.}\end{aligned}$$

To compute the final term, we assume that (F_i) has been chosen as in the proof of Proposition 2.1, and then note that for all $X \in TM$ we have:

$$\text{tr } \nabla_X h = \langle \nabla_X h(F_i), F_i \rangle = \langle \nabla_X(hF_i), F_i \rangle = X \cdot \langle hF_i, F_i \rangle = X \cdot \text{tr } h = 0, \quad (2.21)$$

since $h\xi = 0$ and h is trace-free. Hence:

$$\begin{aligned} \langle F_i, \nabla_{JZ}(\phi h)JF_i \rangle &= -\langle F_i, \nabla_{JZ}(h\phi)JF_i \rangle = \langle F_i, \nabla_{JZ}h(F_i) - h\nabla_{JZ}\phi(JF_i) \rangle \\ &= \langle F_i, h\bar{\nabla}_Z J(F_i) \rangle, \quad \text{by (2.21), (2.6) and Lemma 2.1,} \\ &= \langle h, \bar{\nabla}_Z J \rangle = \langle h, \nabla_Z \phi \rangle = 0, \end{aligned}$$

since h (resp. $\nabla_Z \phi$) is symmetric (resp. skew-symmetric). \square

Theorem 2.2. *An H -contact structure on M is harmonic if and only if the \ast Ricci curvature of M is symmetric.*

As an example, we consider the (κ, μ) -manifolds introduced in [4]. These are the contact metric manifolds whose curvature satisfies:

$$R(X, Y)\xi = (\kappa + \mu h)r(X, Y)\xi, \quad \text{for all } X, Y \in TM, \quad (2.22)$$

where κ, μ are constants, and r is the curvature-type tensor defined in (2.17). It was shown in [9] that (κ, μ) -manifolds are H -contact. Furthermore in [4] it was shown that (2.22) determines R completely (for a contact metric manifold), which enables us to analyse the \ast Ricci curvature.

Theorem 2.3. *A contact metric structure satisfying the (κ, μ) -nullity condition is harmonic.*

Proof. We note first that since $r(\mathcal{D}, \mathcal{D})\xi = 0$ it follows from (2.22) that $\rho^*(\xi, Z) = 0$ for all $Z \in \mathcal{D}$; therefore $\rho^*(\xi, Z) = \rho^*(Z, \xi)$. Now for all $Z, W \in \mathcal{D}$ we have the identity:

$$2\rho^*(Z, W) - 2\rho^*(W, Z) = \langle [R(F_i, JF_i), J](JZ), JW \rangle,$$

which follows from (2.19). The curvature identity [4, Lemma 3.2] may be recast in the following succinct way:

$$\begin{aligned} \langle [R(X, Y), \phi]Z, W \rangle &= \langle [(1+h)r(X, Y)(1+h), \phi]Z, W \rangle \\ &\quad + (1-\kappa)\langle r(X, Y)\xi, \phi r(Z, W)\xi \rangle + (1-\mu)\langle hr(X, Y)\xi, \phi r(Z, W)\xi \rangle, \end{aligned}$$

from which it follows that:

$$\langle [R(F_i, JF_i), J](JZ), JW \rangle = \langle [(1+h)r(F_i, JF_i)(1+h), J](JZ), JW \rangle.$$

Now, using the anticommutativity of J and h :

$$\begin{aligned} [(1+h)r(F_i, JF_i)(1+h), J] &= (1+h)r(F_i, JF_i)(1+h)J - J(1+h)r(F_i, JF_i)(1+h) \\ &= (1+h)(1-h) - (1+h)J(1+h)J + J(1+h)J(1+h) + J(1+h)J(1+h) \\ &= 2(1+h)(1-h) - 2(1-h)(1+h) = 0. \end{aligned}$$

Therefore ρ^* is symmetric, and the result follows from Theorem 2.2. \square

3. SUBMERSIVE ALMOST CONTACT STRUCTURES

Initially let M^{2n+1} be an almost contact metric manifold. We say that the almost contact metric structure is *submersive* if there exists an almost Hermitian manifold $(\hat{M}^{2n}, \hat{g}, \hat{J})$ and a Riemannian submersion $\pi: (M, g) \rightarrow (\hat{M}, \hat{g})$ such that the almost complex structures in \mathcal{D} and $T\hat{M}$ are compatible:

$$d\pi(JZ) = \hat{J}d\pi(Z), \quad \text{for all } Z \in \mathcal{D}. \quad (3.1)$$

We refer to \hat{J} as the *projected* almost Hermitian structure, and henceforward, for simplicity, usually make no notational distinction between J and \hat{J} , denoting the latter by J . We also denote both g and \hat{g} by $\langle \cdot, \cdot \rangle$. Mixing the terminology of almost contact geometry and Riemannian submersions [8], tangent vectors to M in the direction of ξ are *vertical*, whereas elements of \mathcal{D} are *horizontal*. Recall also that a vector field X on M is said to be *basic* if and only if X is horizontal and π -related to a vector field \hat{X} on \hat{M} : $\pi_*X = \hat{X}$. We will denote by \mathcal{V} and \mathcal{H} the orthogonal projections of TM onto the vertical and horizontal subbundles. Thus, for all $X \in TM$:

$$\mathcal{V}X = \eta(X)\xi, \quad \text{and} \quad \mathcal{H}X = X_{\mathcal{D}} = \bar{X}, \quad \text{where} \quad X = \bar{X} + \eta(X)\xi.$$

We will utilize O'Neill's structure tensor A , defined:

$$A_X Y = \mathcal{V}(\nabla_{\mathcal{H}X} \mathcal{H}Y) + \mathcal{H}(\nabla_{\mathcal{H}X} \mathcal{V}Y), \quad \text{for all } X, Y \in TM.$$

Finally we recall Lemma 2 and Theorem 2 of [8], which in our context become:

Lemma 3.1. *If X and Y are sections of \mathcal{D} , then*

$$A_X Y = \frac{1}{2} \eta([X, Y]) \xi.$$

Lemma 3.2. *If X, Y, Z, H are elements of \mathcal{D} , then:*

$$\begin{aligned} g(R(X, Y)Z, H) &= \hat{g}(\hat{R}(\hat{X}, \hat{Y})\hat{Z}, \hat{H}) - 2g(A_X Y, A_Z H) \\ &\quad + g(A_Y Z, A_X H) + g(A_Z X, A_Y H). \end{aligned}$$

Now, if M is a compact regular K -contact manifold, then M is submersive, via the Boothby-Wang fibration [5]. Conversely, suppose that M is a submersive contact metric manifold. Suppose $X \in \mathcal{D}$ is extended to a local basic field. Then ϕX is also basic, by (3.1), and therefore:

$$\begin{aligned} 2\pi_*(hX) &= \pi_* L_\xi \phi(X) = \pi_* [\xi, \phi X] - \pi_* \phi[\xi, X] \\ &= [\pi_* \xi, \pi_* \phi X] - J\pi_* [\xi, X], \quad \text{by (3.1),} \\ &= -J[\pi_* \xi, \pi_* X] = 0. \end{aligned}$$

Since h is \mathcal{D} -valued (on a contact metric manifold) it follows that $h = 0$. Thus M is necessarily K -contact, and \hat{M} is therefore almost Kähler.

Theorem 3.1. *A submersive contact metric structure is harmonic if and only if the projected almost Hermitian structure is harmonic.*

Proof. First, since the contact metric manifold is necessarily K -contact ($h = 0$), it follows from Lemma 3.1 that for all horizontal tangent vectors Z, W we have:

$$\begin{aligned} 2A_Z W &= \langle \nabla_X W - \nabla_W Z, \xi \rangle \xi = -\langle W, \nabla_Z \xi \rangle \xi + \langle X, \nabla_Z \xi \rangle \xi \\ &= \langle W, JZ \rangle \xi - \langle Z, JW \rangle \xi = 2\langle W, JZ \rangle \xi, \quad \text{by (2.3).} \end{aligned}$$

Therefore:

$$A_Z W = \langle JZ, W \rangle \xi. \tag{3.2}$$

Now, since $h = 0$, it follows from (2.20) that for all $X, Y \in \mathcal{D}$ we have:

$$\bar{\rho}^*(X, Y) = \langle R(X, F_i)JF_i, JY \rangle + (2n - 1)\langle X, Y \rangle,$$

and then from Lemma 3.2:

$$\begin{aligned} &= \langle \hat{R}(\hat{X}, \hat{F}_i) J \hat{F}_i, J \hat{Y} \rangle - 2 \langle A_X F_i, A_{J F_i} J Y \rangle \\ &\quad + \langle A_{F_i} J F_i, A_X J Y \rangle + \langle A_{J F_i} X, A_{F_i} J Y \rangle + (2n - 1) \langle X, Y \rangle, \end{aligned}$$

and finally using (3.2):

$$\begin{aligned} &= \hat{\rho}^*(\hat{X}, \hat{Y}) - 2 \langle J X, F_i \rangle \langle J^2 F_i, J Y \rangle \\ &\quad + \langle J F_i, J F_i \rangle \langle J X, J Y \rangle + \langle J^2 F_i, X \rangle \langle J F_i, J Y \rangle + (2n - 1) \langle X, Y \rangle \\ &= \hat{\rho}^*(\hat{X}, \hat{Y}) + 4n \langle X, Y \rangle. \end{aligned}$$

Therefore the \ast Ricci curvature of \mathcal{D} is symmetric if and only if the \ast Ricci curvature $\hat{\rho}^*$ of \hat{M} is symmetric, which by [13] is equivalent to the almost complex structure of \hat{M} being harmonic, since \hat{M} is an almost Kähler manifold. The result then follows from Theorem 2.1. \square

There are of course submersive almost contact metric structures which are not contact. As an example, we consider the warped product $M = \hat{M} \times_f \mathbb{R}$, where $f: \hat{M} \rightarrow \mathbb{R}$ is a strictly positive function. The induced almost contact metric structure is determined by the stipulation that $\xi = f^{-1} \partial_t$ (where for convenience we are identifying $f \circ \pi$ with f) and $\mathcal{D} = T\hat{M}$ equipped with the almost Hermitian structure of \hat{M} . In order to analyse the harmonic section equations we recall the following elementary aspects of warped product geometry [1, Lemma 7.3].

Lemma 3.3. *Let M be the warped product $\hat{M} \times_f \check{M}$. Let X, Y be projectable horizontal (ie. basic) vector fields, and V, W projectable vertical vector fields. Then:*

- (1) $\nabla_X V = \nabla_V X = \langle X, f^{-1} \nabla f \rangle V$;
- (2) $\mathcal{H}(\nabla_V W) = -\langle V, W \rangle f^{-1} \nabla f$ and $\mathcal{V}(\nabla_V W) = \check{\nabla}_V W$;
- (3) $\nabla_X Y = \hat{\nabla}_X Y$.

Here \langle, \rangle denotes the warped metric, and projectable tangent vectors on M are identified with their projections to \hat{M} or \check{M} as appropriate. Furthermore ∇f denotes the horizontal lift of the gradient field of f , or equivalently the gradient of $f \circ \pi$.

Proposition 3.1. *On the warped product almost contact manifold $\hat{M} \times_f \mathbb{R}$ the Reeb field is harmonic, and the first harmonic section equation is verified.*

Proof. First we show that the rough Laplacian of the Reeb field is:

$$\nabla^* \nabla \xi = f^{-2} |\nabla f|^2 \xi, \quad (3.3)$$

so ξ is a harmonic vector field. To see this, suppose $X \in \mathcal{D}_x$ has been extended to a local basic vector field with $\nabla X(x) = 0$, which is possible by Lemma 3.3. Then by Lemma 3.3:

$$\nabla_X \xi = (X.f^{-1})\partial_t + f^{-1} \nabla_X \partial_t = -f^{-2} (X.f) \partial_t + f^{-1} (f^{-1} (X.f)) \partial_t = 0. \quad (3.4)$$

This implies that $\nabla_{X,X}^2 \xi = 0$. On the other hand, using that f does not depend on t , and ∂_t is a unit vector field on \mathbb{R} with geodesic integral curves, it follows from Lemma 3.3 that:

$$\nabla_\xi \xi = f^{-2} \nabla_{\partial_t} \partial_t = -f^{-3} \langle \partial_t, \partial_t \rangle \nabla f = -f^{-1} \nabla f, \quad (3.5)$$

and then:

$$\begin{aligned} \nabla_{\xi,\xi}^2 \xi &= \nabla_\xi \nabla_\xi \xi - \nabla_{\nabla_\xi \xi} \xi = -\nabla_\xi (f^{-1} \nabla f) + f^{-1} \nabla_{\nabla f} (f^{-1} \partial_t) \\ &= f^{-2} (\nabla_{\nabla f} \partial_t - \nabla_{\partial_t} (\nabla f) - |\nabla f|^2 \xi) = -f^{-2} |\nabla f|^2 \xi, \quad \text{by Lemma 3.3.} \end{aligned}$$

This establishes (3.3), and hence $\tau(\xi) = 0$. We now compute:

$$\phi \nabla_\xi \phi (\nabla_\xi \xi) = -f^{-1} \phi \nabla_\xi \phi (\nabla f) = -f^{-1} \phi (\nabla_\xi (\phi \nabla f) - \phi \nabla_\xi (\nabla f)) = 0,$$

since $\nabla_\xi (\nabla f)$ and $\nabla_\xi (\phi \nabla f)$ are vertical, by Lemma 3.3. It follows from (2.6) and (3.4) that $T(\phi) = 0$. Therefore (1.1) is verified. \square

Since we are no longer in a position to use the curvature results of §2, in order to analyse the second harmonic section equation we begin with the following general observation.

Lemma 3.4. *For any submersive almost contact metric structure, if X, Y are elements of \mathcal{D} then:*

$$\pi_* \bar{\nabla}_X J(Y) = \hat{\nabla}_{\hat{X}} J(\hat{Y}).$$

Proof. We extend Y to a basic vector field. It follows from (3.1) that JY is also basic, with $\widehat{JY} = J\hat{Y}$. Furthermore, by [8, Lemma 1]:

$$\pi_* \nabla_X Y = \hat{\nabla}_{\hat{X}} \hat{Y}. \quad (3.6)$$

Therefore:

$$\begin{aligned} \pi_*(\bar{\nabla}_X J(Y)) &= \pi_* \bar{\nabla}_X (JY) - \pi_* J \bar{\nabla}_X Y = \hat{\nabla}_{\hat{X}} \widehat{JY} - J \pi_* \bar{\nabla}_X Y, \quad \text{by (3.1)} \\ &= \hat{\nabla}_{\hat{X}} (J\hat{Y}) - J \hat{\nabla}_{\hat{X}} \hat{Y} = \hat{\nabla}_{\hat{X}} J(\hat{Y}). \end{aligned} \quad \square$$

Proposition 3.2. *On the warped product almost contact manifold $\hat{M} \times_f \mathbb{R}$ the second harmonic section equation is verified if and only if the almost Hermitian structure on \hat{M} satisfies:*

$$[\hat{\nabla}^* \hat{\nabla} J, J] + 2f^{-1} J \hat{\nabla}_{\nabla f} J = 0.$$

Proof. We note first that if $Z \in \mathcal{D}_x$ is extended to a local basic field then since ϕZ is also basic it follows from Lemma 3.3 that:

$$\nabla_\xi \phi(Z) = \nabla_\xi (\phi Z) - \phi \nabla_\xi Z = \langle \phi Z, f^{-1} \nabla f \rangle \xi.$$

Therefore by (2.6):

$$\bar{\nabla}_\xi J = 0. \quad (3.7)$$

It then follows from (3.5) that:

$$\bar{\nabla}_{\xi, \xi}^2 J = -\bar{\nabla}_{\nabla \xi} J = f^{-1} \bar{\nabla}_{\nabla f} J,$$

and hence by Lemma 3.4:

$$\pi_* \bar{\nabla}_{\xi, \xi}^2 J = f^{-1} \hat{\nabla}_{\nabla f} J. \quad (3.8)$$

Now suppose that $X, Y \in \mathcal{D}_x$ are also extended to local basic fields. It then follows from Lemma 3.4 that, for example, $\bar{\nabla}_Y J(Z)$ is also basic. Let us further suppose that Y, Z have been extended such that $\nabla Y(x) = 0 = \nabla Z(x)$ (cf. the proof of Proposition 3.1). Then at x :

$$\begin{aligned} \pi_* \bar{\nabla}_{X, Y}^2 J(Z) &= \pi_* \bar{\nabla}_X \bar{\nabla}_Y J(Z) = \pi_* \bar{\nabla}_X (\bar{\nabla}_Y J(Z)) \\ &= \hat{\nabla}_{\hat{X}} (\hat{\nabla}_{\hat{Y}} J(\hat{Z})), \quad \text{by (3.6) and Lemma 3.4,} \\ &= \hat{\nabla}_{\hat{X}, \hat{Y}}^2 J(\hat{Z}), \quad \text{by Lemma 3.3.} \end{aligned} \quad (3.9)$$

Combining (3.8) and (3.9) yields:

$$\pi_*[\bar{\nabla}^*\bar{\nabla}J, J] = [\hat{\nabla}^*\hat{\nabla}J, J] + 2f^{-1}J\hat{\nabla}_{\nabla f}J,$$

and the result follows on comparison with (1.2). \square

It is perhaps worth noting that in contrast to Lemma 3.4, the attractive equation (3.9) does not hold for arbitrary submersive almost contact structures; it is in fact a consequence of J being parallel along the Reeb field. We also note that if \hat{M} is $(1, 2)$ -symplectic then it follows from Lemma 3.4 and (3.7) that $\bar{\nabla}J$ has the same symmetries as those of a contact metric structure (Lemma 2.1 and (2.9)), although M is no longer a contact manifold.

Combining Propositions 3.1 and 3.2 gives our second main result on submersive almost contact metric structures.

Theorem 3.2. *The induced almost contact structure on $M = \hat{M} \times_f \mathbb{R}$ is harmonic if and only if the almost Hermitian structure on \hat{M} satisfies:*

$$[\hat{\nabla}^*\hat{\nabla}J, J] + 2f^{-1}J\hat{\nabla}_{\nabla f}J = 0.$$

If \hat{M} is a Kähler manifold then the almost contact structure on M is harmonic for all warping functions f . If the almost Hermitian structure on \hat{M} is harmonic (for example, if \hat{M} is a nearly Kähler manifold) then the almost contact structure on M is harmonic if and only if ∇f is Kähler null.

It follows from Theorem 3.2 that if \hat{M} is a *strict* nearly Kähler manifold [7] (for example, the six dimensional sphere) then the warped product almost contact structure on $\hat{M} \times_f \mathbb{R}$ is never harmonic. We conclude by showing that this situation can be corrected if the warping is reversed, so that $M = \mathbb{R} \times_f \check{M}$ where $f: \mathbb{R} \rightarrow \mathbb{R}$ is strictly positive and \check{M}^{2n} is an almost Hermitian manifold. In this case the induced almost contact metric structure is specified by $\xi = \partial_t$, and $\mathcal{D} = T\check{M}$ equipped with the almost Hermitian structure of \check{M} . Notice however that the almost contact structure is no longer submersive, and ξ is now horizontal, whereas \mathcal{D} is vertical. It follows from Lemma 3.3 that ξ is geodesic.

Lemma 3.5. *The induced almost contact structure on $\mathbb{R} \times_f \check{M}$ verifies $\nabla_\xi \phi = 0$, and:*

$$\bar{\nabla}_X J(Y) = \check{\nabla}_X J(Y), \quad \text{for all } X, Y \in \mathcal{D}.$$

(All projections have been notationally omitted.)

Proof. Since ξ is geodesic, we have $\nabla \phi(\xi, \xi) = 0$. Furthermore if $Y \in \mathcal{D}$ is extended to a projectable vertical field then ϕY is also projectable vertical and so by Lemma 3.3:

$$\nabla_\xi \phi(Y) = \nabla_\xi(\phi Y) - \phi \nabla_\xi Y = f^{-1} f' \phi Y - \phi(f^{-1} f' Y) = 0.$$

Now by (2.6) and Lemma 3.3, taking into account the identification of \mathcal{D} and $T\check{M}$:

$$\begin{aligned} \bar{\nabla}_X J(Y) &= \mathcal{V} \nabla_X \phi(Y) = \mathcal{V} \nabla_X(\phi Y) - \phi \mathcal{V} \nabla_X Y \\ &= \check{\nabla}_X(JY) - J \check{\nabla}_X Y = \check{\nabla}_X J(Y). \end{aligned} \quad \square$$

For our next result, recall that an almost Hermitian manifold is said to be *cosymplectic* if its Kähler form is co-closed. The class of such manifolds includes all $(1, 2)$ -symplectic manifolds, and hence all almost Kähler and nearly Kähler manifolds.

Proposition 3.3. *On the warped product almost contact manifold $\mathbb{R} \times_f \check{M}$ the Reeb field is harmonic, and the first harmonic section equation is verified if and only if \check{M} is a cosymplectic almost Hermitian manifold or f is constant.*

Proof. First we prove that the vector field ξ is harmonic. Since ξ is geodesic, we have $\nabla_{\xi, \xi}^2 \xi = 0$. Now if $X \in \mathcal{D}_x$ is extended to a projectable vertical field with $\check{\nabla} X(x) = 0$ then it follows from Lemma 3.3 that:

$$\nabla_X X = -f^{-1} f' |X|^2 \xi, \quad \text{and} \quad \nabla_X \xi = f^{-1} f' X, \quad (3.10)$$

noting that $\nabla f = f' \xi$. Therefore, since ξ is geodesic:

$$\begin{aligned} \nabla_{X, X}^2 \xi &= \nabla_X \nabla_X \xi = \nabla_X(f^{-1} f' X) = X.(f^{-1} f') X + f^{-1} f' \nabla_X X \\ &= -f^{-2} (f')^2 |X|^2 \xi. \end{aligned}$$

It therefore follows that:

$$\nabla^* \nabla \xi = 2n f^{-2} (f')^2 \xi,$$

and hence ξ is harmonic. Next, we prove that $T(\phi)$ is identically zero and the result will follow. Let (F_i) be a projectable vertical local orthonormal frame. Then since ξ is geodesic:

$$\begin{aligned} T(\phi) &= \bar{\nabla}_{F_i}(\nabla_{F_i}\xi) = f^{-1}f'\bar{\nabla}_{F_i}J(F_i), \quad \text{by (3.10),} \\ &= f^{-1}f'\check{\nabla}_{F_i}J(F_i), \quad \text{by Lemma 3.5,} \\ &= f^{-1}f'\check{\delta}J = 0, \end{aligned}$$

if f is constant or \check{M} is cosymplectic. \square

Theorem 3.3. *Suppose that \check{M} is a cosymplectic almost Hermitian manifold, or f is constant. Then the induced almost contact structure on $\mathbb{R} \times_f \check{M}$ is harmonic if and only if the almost Hermitian structure on \check{M} is harmonic. In particular, if \check{M} is a nearly Kähler manifold then the induced almost contact structure is harmonic.*

Proof. We note first that by Lemma 3.5 and (2.6):

$$\bar{\nabla}_\xi J = 0. \tag{3.11}$$

Therefore since ξ is geodesic:

$$\bar{\nabla}_{\xi,\xi}^2 J = \bar{\nabla}_\xi \bar{\nabla}_\xi J = 0.$$

Now suppose X is as in the proof of Proposition 3.3. Then:

$$\begin{aligned} \bar{\nabla}_{X,X}^2 J &= \bar{\nabla}_X \bar{\nabla}_X J - \bar{\nabla}_{\nabla_X X} J \\ &= \bar{\nabla}_X \bar{\nabla}_X J, \quad \text{by (3.10) and (3.11),} \\ &= \mathcal{V} \nabla_X \check{\nabla}_X J, \quad \text{by Lemma 3.5,} \\ &= \check{\nabla}_X \check{\nabla}_X J, \quad \text{by Lemma 3.3,} \\ &= \check{\nabla}_{X,X}^2 J, \quad \text{since } \check{\nabla} X(x) = 0. \end{aligned}$$

It follows that:

$$\bar{\nabla}^* \bar{\nabla} J = \check{\nabla}^* \check{\nabla} J,$$

and hence (1.2) is verified precisely when the almost Hermitian structure on \check{M} is harmonic. Combining this with Proposition 3.3 yields the result. \square

Inspecting equation (3.11) and Lemma 3.5 shows that once again we are able to construct examples of almost contact structures where $\bar{\nabla} J$ has the same symmetries as those of a contact metric manifold.

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